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THE MOTION OF A HEAVY SYMMETRICAL BODY WITH FLEXIBLE RODS ABOUT A FIXED POINT *

V.G. VIL'KE

The motion of a symmetrical solid about its centre of mass is considered in the case, when four mutually orthogonal flexible rods are fixed to it in the equatorial plane of the body ellipsoid of inertia. The deformations of rods is defined by the linear theory of the bending of thin viscoelastic rods, and lead to the evolution of the motion of the solid, i.e. the solid approaches steady rotation about the vertical. the approximate equations in Andoyer variables that define the system evolution are obtained by the method of averaging. The stability of the steady rotations obtained is investigated.

The stability of steady rotations of a solid with a single fixed point and with flexible rods attached to it was investigated in /1, 2/. It was shown in /3/ that the longitudinal deformations of elastic rods fixed to a heavy symmetrical solid rotating about a fixed point results in the body approaching a steady rotation about the vertical axis. In that paper an approximate equation was also obtained, which defined the evolution of motion in terms of the Andoyer variables by the method of averaging.

Let $A_1 = B_1 \neq C_1$, where (A_1, B_1, C_1 are the principal central moments of inertia of the solid about the point O (the centre of mass of the body), and let two paris of elastic rods be positioned along the principal axes of the ellipsoid of inertia Ox_1 and Ox_2. Using the linear theory of the bending of thin rectilinear rods, we determine the radius vector of a point of the rod in the system of coordinates $Ox_1x_2x_3$ in the form

$$\mathbf{R}_1 = s\mathbf{e}_1 + \mathbf{u}_1 = s\mathbf{e}_1 + u_{12}(s,t)\mathbf{e}_2 + u_{13}(s,t)\mathbf{e}_3$$
$$\mathbf{R}_2 = s\mathbf{e}_2 + \mathbf{u}_2 = u_{21}(s,t)\mathbf{e}_1 + s\mathbf{e}_2 + u_{23}(s,t)\mathbf{e}_3$$
$$s \in K = [-b, a] \cup [a, b]$$

The kinetic energy and angular momentum of the system are defined by the relations

$$T = \frac{1}{2}(J_1\boldsymbol{\omega}, \boldsymbol{\omega}) + \frac{1}{2}\sum_{i=1}^{2}\int_K [(\boldsymbol{\omega}\times\mathbf{R}_i) + \mathbf{R}_i^{\cdot}]^2 \rho\, ds$$

$$\mathbf{G} = J_1\boldsymbol{\omega} + \sum_{i=1}^{2}\int_K [\mathbf{R}_i \times (\boldsymbol{\omega}\times\mathbf{R}_i + \mathbf{R}_i^{\cdot})]\rho\, ds$$

where $\boldsymbol{\omega}(\omega_1, \omega_2, \omega_3{}^*)$ is the angular velocity of rotation of the body, J_1 is the inertia tensor of the body, and ρ is the linear density of the rod material, which is assumed homogeneous. The angular velocity and the inertia tensor are considered in the moving system of coordinates $Ox_1x_2x_3$.

The position of the moving coordinate system relative to the fixed system $O\xi_1\xi_2\xi_3$ (the axis $O\xi_3$ is vertical) is defined by Euler's angles. The generalized momenta and Routh's functional are defined by the relations

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$$p_\varphi = \mathbf{G}\mathbf{e}_\varphi, \quad p_\psi = \mathbf{G}\mathbf{e}_\psi, \quad p_\theta = \mathbf{G}\mathbf{e}_\theta, \quad \mathbf{G} = \nabla_\omega T$$
$$R = p_\varphi \varphi^{\cdot} + p_\psi \psi^{\cdot} + p_\theta \theta^{\cdot} - T + \Pi + E[\mathbf{u}]$$

where φ, ψ, θ are the angles of natural rotation, precession, and nutation, respectively, $\mathbf{e}_\varphi, \mathbf{e}_\psi, \mathbf{e}_\theta$ are the unit vectors along the axes of rotation corresponding to Euler's angles, and Π, $E[\mathbf{u}]$ are the potential energy functionals of the gravitational field and elastic deformations. The Routh functional is numerically equal to

$$R = \frac{1}{2}(J[\mathbf{u}]\boldsymbol{\omega}, \boldsymbol{\omega}) - \frac{1}{2}\int_K (\mathbf{R}_1^{\cdot 2} + \mathbf{R}_2^{\cdot 2})\rho\, ds + \Pi + E[\mathbf{u}]$$

where $J[\mathbf{u}]$ is the inertia tensor of the system consisting of the solid and deformed rods. Note that

$$J[\mathbf{u}]\boldsymbol{\omega} = \mathbf{G} - \mathbf{G}_u, \quad \mathbf{G}_u = \int_K \sum_{i=1}^{2} [\mathbf{R}_i \times \mathbf{R}_i^{\cdot}]\rho\, ds$$
$$R = \frac{1}{2}(\mathbf{G} - \mathbf{G}_u, J^{-1}[\mathbf{u}](\mathbf{G} - \mathbf{G}_u)) - \frac{1}{2}\int_K (\mathbf{u}_1^{\cdot 2} + \mathbf{u}_2^{\cdot 2})\rho\, ds + \Pi + E[\mathbf{u}]$$

We will change from canonical variables $p_\varphi, p_\psi, p_\theta, \varphi, \psi, \theta$ to the Andoyer canonical variables $I_1, I_2, I_3, \varphi_1, \varphi_2, \varphi_3$, using a canonical transformation. We note first that the operator $J^{-1}[\mathbf{u}]$ can be represented in the form

$$J^{-1}[\mathbf{u}] = [J_0(E + J_0^{-1}J_1 + J_0^{-1}J_2)]^{-1} = \tag{1}$$
$$[E - J_0^{-1}J_1 - J_0^{-1}J_2 + (J_0^{-1}J_1 + J_0^{-1}J_2)^2 + \ldots]J_0^{-1}$$
$$E = \operatorname{diag}\{1, 1, 1\}, \; J_0 = \operatorname{diag}\{A, A, C\}$$
$$A = A_1 + \int_K s^2\rho\, ds, \quad C = C_1 + 2\int_K s^2\rho\, ds$$
$$J_1 = (J_{ij}^{(1)}), \quad J_{ii}^{(1)} = 0, \quad i = 1, 2, 3; \quad J_{12}^{(1)} = -\int_K s(u_{12} + u_{21})\rho\, ds$$
$$J_{j3}^{(1)} = -\int_K s u_{j3}\rho\, ds, \quad j = 1, 2; \quad J_{ij}^{(1)} = J_{ji}^{(1)}$$
$$J_2 = (J_{ij}^{(2)}), \quad J_{ij}^{(2)} = J_{ji}^{(2)}, \quad J_{11}^{(2)} = \int_K (u_{12}^2 + u_{13}^2 + u_{23}^2)\rho\, ds$$
$$J_{22}^{(2)} = \int_K (u_{13}^2 + u_{21}^2 + u_{23}^2)\rho ds$$
$$J_{33}^{(2)} = \int_K (u_{12}^2 + u_{21}^2)\rho\, ds, \quad J_{12}^{(2)} = 0, \quad J_{13}^{(2)} = -\int_K u_{21}u_{23}\rho\, ds$$
$$J_{23}^{(2)} = -\int_K u_{12}u_{23}\rho\, ds$$

We shall henceforth assume u_{ij} to be small and restrict (1) to terms that are linear with respect to u_{ij}

$$J^{-1}[\mathbf{u}] \approx J_0^{-1} - J_0^{-1}J_1J_0^{-1}$$

We will represent the angular momentum $\mathbf{G}$ in terms of the Andoyer variables in the form

$$\mathbf{G} = (\sqrt{I_2^2 - I_1^2}\sin\varphi_1, \quad \sqrt{I_2^2 - I_1^2}\cos\varphi_1, I_1)$$

and define the functionals Π and $E[\mathbf{u}]$ by the formulas

$$\Pi = \rho g \int_K (\mathbf{u}_1 + \mathbf{u}_2)\mathbf{e}\, ds$$
$$E[\mathbf{u}] = \frac{1}{2} N \int_K \sum_{i=1, j=1, j\neq i}^{2,3} \left(\frac{\partial^2 u_{ij}}{\partial s^2}\right)^2 ds$$

where $\mathbf{e}(\gamma_1, \gamma_2, \gamma_3)$ is the unit vector along the vertical axis in the system of coordinates $Ox_1x_2x_3$, g is the acceleration due to gravity, and N is the rod bending rigidity, equal to the product of the modulus of elasticity of the rod material and the moment of inertia of the rod cross-section (the rod is assumed to be unformly rigid). The unit vector directional cosines relative to the vertical are expressed in terms of the Andoyer variables in the form /4/

$$\gamma_1 = a\sin\varphi_1 + b\sin\varphi_1\cos\varphi_2 + c\cos\varphi_1\sin\varphi_2 \tag{2}$$
$$\gamma_2 = a\cos\varphi_1 + b\cos\varphi_1\cos\varphi_2 - c\sin\varphi_1\sin\varphi_2$$
$$\gamma_3 = d - e\cos\varphi_2$$

$$a = I_2^{-2} I_3 \sqrt{I_2^2 - I_1^2}, \quad b = I_1 I_2^{-2} \sqrt{I_2^2 - I_3^2}$$
$$c = I_2^{-1} \sqrt{I_2^2 - I_3^2}, \quad d = I_1 I_3 I_2^{-2}$$
$$e = I_2^{-2} \sqrt{(I_2^2 - I_1^2)(I_2^2 - I_3^2)}$$

The system equations of motion

$$\mathbf{I}^{\cdot} = -\nabla_{\varphi} R[\mathbf{I}, \varphi, \mathbf{u}^{\cdot}, \mathbf{u}]$$
$$\varphi^{\cdot} = \nabla_I R[\mathbf{I}, \varphi, \mathbf{u}^{\cdot} \mathbf{u}] \tag{3}$$
$$\frac{d}{dt} \nabla_{u^{\cdot}} R[\mathbf{I}, \varphi, \mathbf{u}^{\cdot}, \mathbf{u}] - \nabla_u R[\mathbf{I}, \varphi, \mathbf{u}^{\cdot}, \mathbf{u}] = -\mathbf{Q}_u$$

contain in the last equation the small parameter $\varepsilon = N^{-1}$. The dissipative forces $\mathbf{Q}_u$ correspond to the linear model of internal viscous friction

$$\mathbf{Q}_u = -\nabla_{u^{\cdot}} D[\mathbf{u}^{\cdot}], \quad D[\mathbf{u}^{\cdot}] = \frac{1}{2} \chi N \int\limits_K \sum_{i=1,\, j=1,\, j \neq i}^{2,\,3} \left(\frac{\partial^5 u_{ij}}{\partial t\, \partial s^4} \right)^2 ds, \ \chi > 0$$

If $\varepsilon = 0$ (the rods are not deformed), then

$$\mathbf{u} \equiv 0, \ R\,[\mathbf{I}, \varphi, 0, 0] = {}^1\!/_2 A^{-1} (I_2^2 - I_1^2) + {}^1\!/_2 C^{-1} I_1^2$$

and the equations of motion take the form

$$I_i = 0, \ i = 1, 2, 3, \ \varphi_1^{\cdot} = (A - C)\, A^{-1} C^{-1} I_1$$
$$\varphi_2^{\cdot} = A^{-1} I_2, \ \varphi_3^{\cdot} = 0 \tag{4}$$

The solutions of (4) define the regular precession of a symmetric solid in the Euler case. The solution of the last of Eqs.(3), when $\mathbf{I} = \mathbf{I}_0$, $\varphi = \varphi^{\cdot}(I_0)\, t + \varphi(0)$ is obtained in the form of the series $u_{ij} = \varepsilon u_{ij}^{(1)} + \varepsilon^2 u_{ij}^{(2)} + \ldots$, $i = 1, 2$, $j = 1, 2, 3$, $j \neq i$, where the functions $u_{ij}^{(1)}$ satisfy the equation

$$\frac{\partial^4 u_{ij}^{(1)}}{\partial s^4} + \chi \frac{\partial^5 u_{ij}^{(1)}}{\partial t\, \partial s^4} = -s\rho\varepsilon\omega_i\omega_j - \rho\varepsilon g\gamma_j, \quad i = 1, 2, \quad j = 1, 2, 3, \quad j \neq i \tag{5}$$
$$\omega_1 = A^{-1} \sqrt{I_2^2 - I_1^2} \sin\varphi_1, \quad \omega_2 = A^{-1} \sqrt{I_2^2 - I_1^2} \cos\varphi_1$$
$$\omega_3 = C^{-1} I_1 + \varphi_1^{\cdot} = A^{-1} C^{-1} (2A - C)\, I_1$$

If $u_{ij0}^{(1)}$ is a solution of (5), then its particular solution, that defines the forced bending oscillations of rods, has the form

$$u_{ij}^{(1)} = \sum_{k=0}^{\infty} (-\chi)^k \frac{\partial^k u_{ij0}^{(1)}}{\partial t^k} \tag{6}$$

Henceforth, assuming $\chi\varphi_1^{\cdot}$ and $\chi\varphi_2^{\cdot}$ to be small, we shall restrict the series (6) to the first two terms, and obtain

$$u_{ij}^{(1)} \approx -\rho\varepsilon\, \{[\omega_i\omega_j - \chi\,(\omega_i\omega_j)^{\cdot}]\, \psi_1(s) + g\,(\gamma_i - \chi\gamma_j^{\cdot})\, \psi_2(s)\} \tag{7}$$

where when $K = [-b, b]$

$$\psi_1(s) = \frac{s^5}{120} - \frac{b^2 s^3}{12} + \frac{b^3 s^2 \operatorname{sign} s}{6}$$
$$\psi_2(s) = \frac{s^4}{24} - \frac{b s^3 \operatorname{sign} s}{6} + \frac{b^2 s^2}{4}, \quad s \in K$$

When determining the functions ψ_1, ψ_2, the kinematic boundary conditions

$$u_{ij0}^{(1)}(0, t) = \frac{\partial}{\partial s} u_{ij0}^{(1)}(0, t) = 0$$

(the conditions of attachement at the origin of coordinates) and the dynamic boundary conditions

$$\frac{\partial^2}{\partial s^2} u_{ij0}^{(1)}(\pm b, t) = \frac{\partial^3}{\partial s^3} u_{ij0}^{(1)}(\pm b, t) = 0$$

(no external forces at the rod free ends) were taken into account. Note that the dynamic boundary conditions are contained in the last equation of system (3).

For variables I_1, I_2, I_3 Eqs.(3) have the form

$$I_1^{\cdot} = -\left(J^{-1}[\mathbf{u}] \frac{\partial \mathbf{G}}{\partial \varphi_1}, \mathbf{G} - \mathbf{G}_u \right) - \int\limits_K \rho g \left(u_{12} \frac{\partial \gamma_2}{\partial \varphi_1} + u_{21} \frac{\partial \gamma_1}{\partial \varphi_1} \right) ds \tag{8}$$
$$I_2^{\cdot} = -\int\limits_K \rho g \left[u_{21} \frac{\partial \gamma_1}{\partial \varphi_2} + u_{12} \frac{\partial \gamma_2}{\partial \varphi_2} + (u_{13} + u_{23}) \frac{\partial \gamma_3}{\partial \varphi_2} \right] ds, \quad I_3^{\cdot} = 0$$

To determine the evolution of the action variables it is necessary to substitute solutions (7), restricted to terms of the order of ε, into Eqs.(8), and average over the angle variables φ_1, φ_2 /5/. We then have

$$I_1^{\cdot} = -(A-C) I_1 (I_2^2 - I_1^2)[k_1 (I_2^2 - I_1^2 + (2A-C) C^{-1} I_1^2) + k_2 I_3^2 I_2^{-4}] - {}^1/_2 k_2 I_1 (I_2^2 - I_3^2)[A(I_1^2 + I_2^2) + C(I_2^2 - I_1^2)] I_2^{-4} \tag{9}$$

$$I_2^{\cdot} = -k_2 (I_2^2 - I_3^2) I_2^{-3} [{}^3/_2 C (I_2^2 - I_1^2) + A I_1^2], \quad I_3^{\cdot} = 0$$

$$k_1 = \rho^2 \varepsilon \chi A^{-5} C^{-1} \int_K s\psi_1(s)\, ds > 0, \quad k_2 = \rho^2 \varepsilon \chi g^2 A^{-1} C^{-1} \int_K \psi_2(s)\, ds > 0$$

From the third of Eqs.(9) it follows that the projection of the angular momentum vector of the system on the vertical I_3 remains constant, and from the second that the angular momentum of the system I_2 approaches I_3, i.e. the body approaches steady rotation about the vertical.

Let us determine the limit values of I_1. Equating to zero the right side of the first of Eqs.(9) and assuming that $I_2 = I_3$, we obtain the equation

$$(A - C) I_1 (I_2^2 - I_1^2)[k_1 (I_2^2 - I_1^2 + (2A - C) C^{-1} I_1^2) + k_2 I_2^{-2}] = 0$$

which shows that $I_1 = 0, I_1 = I_2$ are the steady values. The frst of them is steady when $A > C$ and unsteady when $A < C$, and, conversely, the second is stable when $A < C$ and unstable when $A > C$.

To follow the evolution of the angular momentum vector, we will obtain an equation describing the variation of the angle φ_3. We have

$$\varphi_3^{\cdot} = \rho g \int_K \left[u_{12} \frac{\partial \gamma_2}{\partial I_3} + u_{21} \frac{\partial \gamma_1}{\partial I_3} + (u_{13} + u_{23}) \frac{\partial \gamma_3}{\partial I_3} \right] ds \tag{10}$$

Taking (2) and (7) into account, after averaging the right side of (10), we obtain

$$\varphi_3^{\cdot} = k_3 I_3 I_2^{-4} (I_2^2 - 3I_1^2), \quad k_3 = {}^1/_2 \rho^2 g^2 \varepsilon \int_K \psi_2(s)\, ds > 0 \tag{11}$$

The angle $\varphi_3(t)$ is obtained by simple integration, after integrating Eqs.(9).

It follows from (11) that the end of vector $\mathbf{G}$ describes in the fixed horizontal plane a helix-like curve. Since a change in the direction of rotation is possible during the motion, the quantity $I_2^2 - 3I_1^2$ may change its sign.

Let us determine the position of the fixed axes of rotation in the equatorial plane of the ellipsoid of inertia, when $A > C$ and I_1 approaches zero. For small I_1 averaging over the angle φ_1 becomes inadmissible. Averaging the first of Eqs.(8) only over the angle φ_2 and assuming that $I_2 = I_3$, we obtain the equation

$$AC(A - C)^{-1} \varphi_1^{\cdot\cdot} = k_4 I_2^4 \sin 4\varphi_1 - k_5 I_2^2 \varphi_1^{\cdot} \cos^2 2\varphi_1 \tag{12}$$

$$k_4 = \frac{1}{4} \rho^2 \varepsilon A^{-4} \int_K s\psi_1(s)\, ds > 0, \quad k_5 = \chi \rho^2 \varepsilon A^{-2} \int_K s\psi_1(s)\, ds > 0$$

which is accurate to terms linear respect to $\varphi_1^{\cdot}, \varphi_1^{\cdot\cdot}$.

The equilibrium positions given by (12) are

$$\varphi_1 = \{{}^1/_2 \pi m\}, \quad \varphi_1 = \{{}^1/_4 \pi + {}^1/_2 \pi m\}, \quad m = 0, \pm 1, \ldots$$

The first series of solutions corresponding to unstable steady rotations about the Ox_1 and Ox_2 axes, while the second represents stable steady rotations about axes rotated by an angle $\pi/4$ relative to the first. When rotating about the axes Ox_1, Ox_2, the rods are straight, and when rotating around axes turned relative to Ox_1, Ox_2 by an angle $\pi/4$ they are bent by the action of centrifugal inertia forces in the system of coordinates $Ox_1x_2x_3$. The nature of the stability of the positions of equilibrium of the rods that have only longitudinal deformations was directly opposite /3/.

Note that the conclusions on the stability of steady rotations were obtained on the basis of Eq.(12) by the method of motion separation and averaging. It would be desirable to check them by investigating the properties of stationary points of variation of the potential energy /6/.

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ON THE CONTROLLED ROTATION OF A SYSTEM OF TWO RIGID BODIES WITH ELASTIC ELEMENTS*

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The problem of controlling the plane rotational motions of two rigid bodies connected by an elastic rod is studied. One end of the rod is attached to the support by a hinge with a spring, the latter modelling the elastic compliance of the fastening, and the other end is rigidly joined to the load. The Hamilton principle is used to obtain the integrodifferential equations and boundary conditions describing the motion of the system support - spring - rod - load. The following problem is posed: it is required to rotate the system by a given angle by means of the controlling force moment, with quenching of the relative oscillations of the load elements which appear as a result of the deformability of the rod and of the elastic torsion of the spring. Similar problem arise in the study of the dynamics and control of the motion of devices used in transporting loads through space (robots, manipulators, load lifting machines, etc.). In computing their control modes a significant part is played not only by the deformability of the elements /1—3/, but also by the elastic compliance of the connecting joints /4/. Asymptotic methods are used to botain a solution of the control problem in question for two limiting cases: 1) the mass of the load carried is much greater than the mass of the rod and support, and 2) the rod has high flexural rigidity. The results obtained represent a development and generalization of the results obtained in /5/. The problems of the dynamics and control of oscillating systems with distributed parameters were investigated using various types of formulation in a number of papers (/5—13/ et. al.).

1. Description of the model and the equations of motion. We consider a mechanical system consisting of two rigid bodies connected by a rod of variable cross-section. The system can execute rotational motions in some plane (Fig.1). One end of the rod is attached to the support G_1 by means of a hinge with a weightless spring, modelling the elastic compliance of the joint. The other end is rigidly fixed to the load G_2, whose linear dimensions are small compared with the length of the rod. The O_1Z_1-axis, perpendicular to the plane of the motion represents the axis of rotation, with respect to which the moment of control forces $M(t)$ is applied. We introduce the $OXYZ$ coordinate system with origin at the centre of the hinge (point O), rotating in the inertial $O_1X_1Y_1Z_1$ space together with the spring and rod. We direct the OX axis along the tangent to the neutral line of the rod at the point O, and the OZ axis along the O_1Z_1 axis of rotation. We assume that the motion of the model is described in the framework of the linear theory of thin rectilinear, inextensible

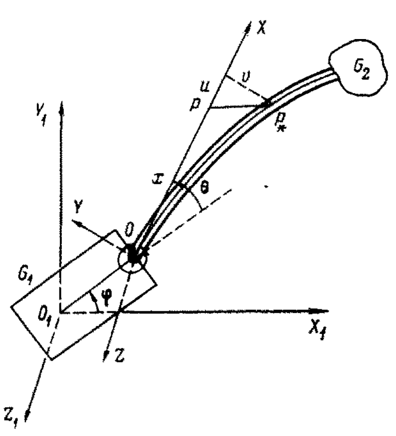

Fig.1

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